

Developing and Applying a Taxonomy for Mathematical Knowledge-Expertise
Jacqueline M. Dewar (jdewar@lmu.edu) and Curtis D. Bennett (cbennett@lmu.edu)
Loyola Marymount University, Los Angeles, CA

A taxonomy for mathematical knowledge-expertise was developed during a year-long study of students' understanding of proof across the undergraduate mathematics major at Loyola Marymount University. The taxonomy takes the form of a matrix with elements adapted from science assessment and expertise theory. Bennett and Dewar (2007) describes how the taxonomy matrix was developed, what lessons it revealed for teaching and learning, and its subsequent application to analyzing student work and faculty instruction

Students' understanding of proof is at the crux of the mathematics major and has been studied by a variety of researchers (see Selden and Selden, 2002; Harel and Sowder, 1998; and Weber, 2001). We used a "proof-aloud" methodology (with 12 students spanning the major and one faculty member serving as expert) around a number theory statement employed in the work of Recio and Godino (2001). The statement was initially presented to the students as something to be investigated and later as a statement to be formally proven. Recio and Godino's rubric proved inadequate to differentiate important aspects of our students' performance on the tasks of generating and writing proofs. In addition, it failed to take into account affective components that influenced the students' performance. In order to fully describe the work of our students, we adapted Shavelson's (2003) typology of scientific knowledge to mathematics and combined it with Alexander's (2003) model of domain learning applied to undergraduates, to produce a taxonomy matrix describing mathematical knowledge and expertise. In both our students' performance and that of our "expert" we found evidence to support the inclusion of a second affective component in addition to Alexander's interest affect. Thus our taxonomy matrix contains two affective (interest and confidence) components and six cognitive (factual, procedural, schematic, strategic, epistemic, and social) components of learning. Across these eight components we have described three stages of expertise: acclimation, competence and proficiency.

In the application of the taxonomy matrix to student work from the proof-alouds we found support for the following hypotheses regarding student learning: Confidence plays an important affective role in student performance; There is a complex relationship between the knowledge components and how students move from acclimation towards proficiency; Even though knowledge and strategies are key to developing expertise, the role of interest in learning and developing expertise is critical as well; Students in acclimation have limited and fragmented knowledge and access to mostly surface-level strategies, and hence they need explicit instruction on what content is central and how to be strategic in their processing.

Although the mathematical knowledge-expertise taxonomy resulted from a study of student understanding of proof, it has much broader applicability to mathematics. After its development we discovered that the mathematical knowledge components of the matrix encompass the five strands of K-8 mathematical proficiency (conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition) delineated in *Adding It Up* (National Research Council, 2001, p. 116). We believe the taxonomy matrix can be viewed as an extension of *Adding It Up's* intertwined strands of elementary school mathematical proficiency (p. 117) to the development of mathematical proficiency in the undergraduate mathematics major. By providing a rich representation of student learning and motivation on the journey toward expertise, the taxonomy matrix for mathematical knowledge-expertise makes possible careful analysis and description of mathematical tasks, students' understandings, and faculty instruction.

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Taxonomy Matrix for Mathematical Knowledge-Expertise

Components of Mathematical Knowledge vs. Stages of Expertise

Curtis Bennett (cbennett@lmu.edu) and Jacqueline Dewar (jdewar@lmu.edu)
Loyola Marymount University, Los Angeles, CA 90045

Affective	Acclimation	Competence	Proficiency
Interest	Students are motivated to learn by external (often grade-oriented) reasons that lack any direct link to the field of study in general. Students have greater interest in concrete problems and special cases than abstract or general results.	Students are motivated by both internal (e.g., intrigued by the problem) and external reasons. Students still prefer concrete concepts to abstractions, even if the abstraction is more useful.	Students have both internal and external motivation. Internal motivation comes from an interest in the problems from the field, not just applications. Students appreciate both concrete and abstract results.
Confidence	Students are unlikely to spend more than 5 minutes on a problem if they cannot solve it. Students don't try a new approach if first approach fails. When given a derivation or proof, they want minor steps explained. They rarely complete problems requiring a combination of steps.	Students spend more time on problems. They will often spend 10 minutes on a problem before quitting and seeking external help. They may consider a second approach. They are more comfortable accepting proofs with some steps "left to the reader" if they have some experience with the missing details. They can start multi-step problems, but may have trouble completing them.	Students will spend a great deal of time on a problem and try more than one approach before going to text or instructor. Students will disbelieve answers in the back of the book if the answer disagrees with something they feel they have done correctly. Students are accustomed to filling in the details of a proof. Can solve multi-step problems.
Cognitive	Acclimation	Competence	Proficiency
Factual	Students start to become aware of basic facts of the topic.	Students have working knowledge of the facts of the topic, but may struggle to access the knowledge.	Students have quick access to and broad knowledge about the topic.
Procedural	Students start to become aware of basic procedures. Can begin to mimic procedures from the text.	Students have working knowledge of the main procedures. Can access them without referencing the text, but may make errors or have difficulty with more complex procedures.	Students can use procedures without reference to external sources or struggle. Students are able to fill in missing steps in procedures.
Schematic	Students begin to combine facts and procedures into packets. They use surface level features to form schema.	Students have working packets of knowledge that tie together ideas with common theme, method, and/or proof.	Students have put knowledge together in packets that correspond to common theme, method, or proof, together with an understanding of the method.
Strategic	Students use surface level features of problems to choose between schema, or they apply the most recent method.	Students choose schema to apply based on just a few heuristic strategies. Students are slow to abandon a non-productive approach.	Students choose schema to apply based on many different heuristic strategies. Students self-monitor and abandon a nonproductive approach for an alternate.
Epistemic	Students begin to understand what constitutes 'evidence' in the field. They begin to recognize that a valid proof cannot have a counterexample. They are likely to believe based on 5 examples; however, they may be skeptical.	Students are more strongly aware that a valid proof cannot have counterexamples. They use examples to decide on the truth of a statement, but require a proof for certainty.	Students recognize that proofs don't have counterexamples, are distrustful of 5 examples, see that general proofs apply to special cases, and are more likely to use "hedging" words to describe statements they suspect to be true but have not yet verified.
Social	Students will struggle to write a proof and include more algebra or computations than words. Only partial sentences will be written, even if they say full sentences. Variables will seldom be defined, and proofs lack logical connectors.	Students are likely to use an informal shorthand that can be read like sentences for writing a proof. They may employ connectors, but writing lacks clarity often due to reliance on pronouns or inappropriate use or lack of mathematical terminology.	Students in this stage write proofs with complete sentences. They use clear concise sentences and employ correct terminology. They use variables correctly.